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Sufficient conditions for convergence of antinormally ordered expansions of boson number operator functions $f(\hat{a}^+\hat{a})$

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Abstract. From previous work on ordering expansions of the boson number operator $\exp(-\mu\hat{a}^+\hat{a})$ formulae are derived for normally and antinormally ordered forms of arbitrary boson number operator functions $f(\hat{a}^+\hat{a})$ in terms of the Fourier transform $g(\lambda)$ of $f(x)$. When the antinormally ordered form $f^{(a)}(\hat{a}^+\hat{a})$ is applied to states belonging to the domain of definition of $f(\hat{a}^+\hat{a})$, the result differs in general from the action of $f(\hat{a}^+\hat{a})$ on these states or is not even defined. For the number operator eigenstates $|m\rangle$ two criteria are derived which give conditions on the Fourier transform $g(\lambda)$ ensuring validity of $f^{(a)}(\hat{a}^+\hat{a})|m\rangle = f(m)|m\rangle$. The general considerations are demonstrated by two examples for functions f.

1. Introduction

Normal- and antinormal-ordering expansions of functions of boson annihilation and creation operators $\hat{a}$, $\hat{a}^+$ are important mathematical tools for the treatment of systems of harmonic oscillators and quantised fields, mainly in quantum optics, especially in laser theory (Lax and Louisell 1967, Haken *et al* 1967, Lax and Yuen 1968) and in studies of coherence properties of light (Sudarshan 1963, Glauber 1963a, b, Mandel and Wolf 1965). In all these fields quantum-statistical averages $\mathrm{Tr}(\hat{\rho}\hat{f})$ of operators $\hat{f}$ have to be determined for systems described by density operators $\hat{\rho}$. The use of ordered operator expansions arises from the fact that $\mathrm{Tr}(\hat{\rho}\hat{f})$ can be converted to an integral over certain c-number functions associated with the normally ordered expansion of $\hat{\rho}$ and with the antinormally ordered expansion of $\hat{f}$ or *vice versa* (for details see e.g. Louisell 1964, 1973). In the quantum theory of optical coherence a more direct physical meaning can be ascribed to the expectation value of normally ordered multiple products of field operators as being proportional to the rate of delayed multiple coincidence counting of photoelectric detectors by absorbing photons (Glauber 1963a, b, Mandel and Wolf 1965). On the other hand, expectation values of antinormally ordered products of field operators can be shown to correspond to correlations of the electromagnetic field measured with the help of quantum counters based on stimulated emission of photons (Mandel 1966).

When an operator-valued function $f(\hat{a}, \hat{a}^+)$ is given by a power series expansion in $\hat{a}$ and $\hat{a}^+$, normal (antinormal) ordering consists in moving all the $\hat{a}$'s to the right (left) of all $\hat{a}^+$'s. Since, in this algebraic procedure, the commutation relation $[\hat{a}, \hat{a}^+] = 1$

has to be taken into account, it is assumed that the operator $f(\hat{a}, \hat{a}^+)$ has the same effect on any boson state $|\psi\rangle$ as has the normally ordered expression

$$f^{(n)}(\hat{a}, \hat{a}^+) \equiv \sum_{n,m=0}^{\infty} c_{nm} (\hat{a}^+)^n \hat{a}^m \tag{1}$$

or the corresponding antinormally ordered expression

$$f^{(a)}(\hat{a}, \hat{a}^+) \equiv \sum_{n,m=0}^{\infty} d_{nm} \hat{a}^m (\hat{a}^+)^n. \tag{2}$$

In other words, we expect

$$f(\hat{a}, \hat{a}^+) = f^{(n)}(\hat{a}, \hat{a}^+) = f^{(a)}(\hat{a}, \hat{a}^+). \tag{3}$$

For certain simple classes of functions f the ordering can be performed to yield closed expressions (Marburger 1966) which have been derived by special techniques like coherent state formalism (Louisell 1973, Mehta 1968, 1977, Gluck 1972), parameter differentiation (McCoy 1932, Louisell 1964, 1973, Schwinger *et al* 1965, Wilcox 1967) or phase-space mapping (Agarwal and Wolf 1970).

In applying these methods it was tacitly assumed that equation (3) is always valid, i.e. that:

(i) the domains $\mathbb{D}$, $\mathbb{D}^{(n)}$, and $\mathbb{D}^{(a)}$ of definition of f, $f^{(n)}$ and $f^{(a)}$, respectively, are all the same;

(ii) for all $|\psi\rangle \in \mathbb{D}$ we have that

$$f|\psi\rangle = f^{(n)}|\psi\rangle = f^{(a)}|\psi\rangle. \tag{3a}$$

However, even for rather elementary, well behaved functions it turned out that (i) is not necessarily satisfied (Cahill and Glauber 1969), especially for the case of antinormal ordering.

For the special case when f depends only on the number operator $\hat{a}^+\hat{a}$, but not on $\hat{a}, \hat{a}^+$ separately, (i) and (ii) can be investigated conveniently if one uses the eigenstate representation $\{|m\rangle\}$ of $\hat{a}^+\hat{a}$, where

$$\hat{a}^+\hat{a}|m\rangle = m|m\rangle \tag{4}$$

($m = 0, 1, 2, \ldots$). It follows that

$$f(\hat{a}^+\hat{a})|m\rangle = f(m)|m\rangle \tag{5}$$

where $f(m)$ is the value of the c-number function $f(x)$ for $x = m$, and equation (3a) now reads

$$f(m)|m\rangle = f^{(n)}(\hat{a}^+\hat{a})|m\rangle = f^{(a)}(\hat{a}^+\hat{a})|m\rangle. \tag{3b}$$

In a recent paper (Baltin 1982) the author has shown for the exponential

$$f(\hat{a}^+\hat{a}) = \exp(-\mu\hat{a}^+\hat{a}) \tag{6}$$

(μ complex) that

$$[\exp(-\mu\hat{a}^+\hat{a})]^{(n)}|m\rangle = \exp(-\mu\hat{a}^+\hat{a})|m\rangle = \mathrm{e}^{-\mu m}|m\rangle \tag{7}$$

for all $|m\rangle$ and all μ, but that

$$[\exp(-\mu\hat{a}^+\hat{a})]^{(a)}|m\rangle = \mathrm{e}^{-\mu m}|m\rangle \tag{8}$$

for all $|m\rangle$ only if

$$|1-e^{\mu}|<1. \tag{9a}$$

For

$$|1-e^{\mu}|\geqslant 1 \tag{9b}$$

$[\exp(-\mu\hat{a}^+\hat{a})]^{(a)}|m\rangle$ has infinite norm for all occupation numbers $m=0,1,2,3,\ldots$, i.e. the antinormally ordered expansion of $\exp(-\mu\hat{a}^+\hat{a})$ does not even exist.

The purpose of the present paper is to study conditions under which antinormal ordering of arbitrary functions $f(\hat{a}^+\hat{a})$ can be achieved such that equation (3*b*) remains valid.

Starting from well known expressions for the normally and antinormally ordered exponential (6) corresponding ordering formulae are established for $f(\hat{a}^+\hat{a})$ in terms of the Fourier transform of the *c*-number function $f(x)$ (§ 2). In § 3 formal expressions for $f^{(n)}(\hat{a}^+\hat{a})|m\rangle$ and $f^{(a)}(\hat{a}^+\hat{a})|m\rangle$ are derived. Using the above-mentioned results on ordering the exponential, two sufficient criteria on consistent antinormal ordering of $f(\hat{a}^+\hat{a})$, i.e. including validity of equation (3*b*), are obtained in § 4. These general considerations are then applied to two operator functions as examples (§ 5). In § 6 results are summarised and discussed.

2. Ordered expressions of arbitrary operator functions $f(\hat{a}^+\hat{a})$ in terms of Fourier transforms

Suppose the existence of the Fourier transform $g(\lambda)$ of the real *c*-number function $f(x)$

$$f(x)=\int_{-\infty}^{+\infty} g(\lambda)\,e^{-i\lambda x}\,d\lambda \tag{10a}$$

or

$$g(\lambda)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)\,e^{i\lambda x}\,dx. \tag{10b}$$

When $f(\hat{a}^+\hat{a})$ is defined by replacing x by $\hat{a}^+\hat{a}$, we obtain

$$f(\hat{a}^+\hat{a})=\int_{-\infty}^{+\infty} g(\lambda)\exp(-i\lambda\hat{a}^+\hat{a})\,d\lambda \tag{11}$$

and hence, at least in a formal manner,

$$[f(\hat{a}^+\hat{a})]^{(n)}=\int_{-\infty}^{+\infty} g(\lambda)[\exp(-i\lambda\hat{a}^+\hat{a})]^{(n)}\,d\lambda \tag{12a}$$

and

$$[f(\hat{a}^+\hat{a})]^{(a)}=\int_{-\infty}^{+\infty} g(\lambda)[\exp(-i\lambda\hat{a}^+\hat{a})]^{(a)}\,d\lambda. \tag{12b}$$

When the well known expressions for ordered exponentials (see e.g. Louisell 1973)

$$(\exp(-\mu\hat{a}^+\hat{a}))^{(n)}=\sum_{\nu=0}^{\infty}\frac{(e^{-\mu}-1)^{\nu}}{\nu!}(\hat{a}^+)^{\nu}\hat{a}^{\nu} \tag{13a}$$

$$(\exp(-\mu\hat{a}^{+}\hat{a}))^{(\mathrm{a})}=\mathrm{e}^{\mu}\sum_{\nu=0}^{\infty}\frac{(1-\mathrm{e}^{\mu})^{\nu}}{\nu!}\hat{a}^{\nu}(\hat{a}^{+})^{\nu} \tag{13b}$$

are inserted in equations (12a), (12b), respectively, putting $\mu=-\mathrm{i}\lambda$, we obtain

$$f^{(\mathrm{n})}(\hat{a}^{+}\hat{a})=\sum_{\nu=0}^{\infty}c_{\nu}(\hat{a}^{+})^{\nu}\hat{a}^{\nu} \tag{14a}$$

where

$$c_{\nu}=\frac{1}{\nu!}\int_{-\infty}^{+\infty}g(\lambda)(\mathrm{e}^{-\mathrm{i}\lambda}-1)^{\nu}\,\mathrm{d}\lambda \tag{15a}$$

and

$$f^{(\mathrm{a})}(\hat{a}^{+}\hat{a})=\sum_{\nu=0}^{\infty}d_{\nu}\hat{a}^{\nu}(\hat{a}^{+})^{\nu} \tag{14b}$$

where

$$d_{\nu}=\frac{1}{\nu!}\int_{-\infty}^{+\infty}g(\lambda)\,\mathrm{e}^{\mathrm{i}\lambda}(1-\mathrm{e}^{\mathrm{i}\lambda})^{\nu}\,\mathrm{d}\lambda. \tag{15b}$$

3. Effect of ordered operator expressions on basis states $|m\rangle$: formal results

Let us apply the operator expansions (14a, b) to the occupation number eigenstate $|m\rangle$. Noting

$$\hat{a}|m\rangle=\sqrt{m}|m-1\rangle \qquad \hat{a}^{+}|m\rangle=\sqrt{m+1}|m+1\rangle \tag{16a, b}$$

we obtain from equations (14a, 15a)

$$\begin{aligned}f^{(\mathrm{n})}(\hat{a}^{+}\hat{a})|m\rangle&=\sum_{\nu=0}^{\infty}c_{\nu}(\hat{a}^{+})^{\nu}\hat{a}^{\nu}|m\rangle\\&=\sum_{\nu=0}^{\infty}\frac{1}{\nu!}\int_{-\infty}^{+\infty}g(\lambda)(\mathrm{e}^{-\mathrm{i}\lambda}-1)^{\nu}\,\mathrm{d}\lambda\, m(m-1)\ldots(m-\nu+1)|m\rangle\\&=|m\rangle\sum_{\nu=0}^{\infty}\binom{m}{\nu}\int_{-\infty}^{+\infty}g(\lambda)(\mathrm{e}^{-\mathrm{i}\lambda}-1)^{\nu}\,\mathrm{d}\lambda.\end{aligned}$$

Since the sum terminates, we are allowed to interchange summation and integration. Using the binomial theorem, we find

$$f^{(\mathrm{n})}(\hat{a}^{+}\hat{a})|m\rangle=|m\rangle\int_{-\infty}^{+\infty}g(\lambda)[1+(\mathrm{e}^{-\mathrm{i}\lambda}-1)]^{m}\,\mathrm{d}\lambda=|m\rangle\int_{-\infty}^{+\infty}g(\lambda)\,\mathrm{e}^{-\mathrm{i}\lambda m}\,\mathrm{d}\lambda$$

or from equation (10a)

$$f^{(\mathrm{n})}(\hat{a}^{+}\hat{a})|m\rangle=f(m)|m\rangle. \tag{17}$$

Thus equation (3b) is always satisfied for normally ordered expressions of $f(\hat{a}^{+}\hat{a})$.

Using (14*b*, 15*b*), and (16*a*, *b*), we obtain for antinormal ordering

$$f^{(a)}(\hat{a}^+\hat{a})|m\rangle = \sum_{\nu=0}^{\infty} d_\nu \hat{a}^\nu (\hat{a}^+)^\nu |m\rangle$$

$$= |m\rangle \sum_{\nu=0}^{\infty} \binom{m+\nu}{\nu} \int_{-\infty}^{+\infty} g(\lambda)\, e^{i\lambda}(1-e^{i\lambda})^\nu \, d\lambda$$

$$= |m\rangle \lim_{n\to\infty} \sum_{\nu=0}^{n} \binom{m+\nu}{\nu} \int_{-\infty}^{+\infty} g(\lambda)\, e^{i\lambda}(1-e^{i\lambda})^\nu \, d\lambda$$

$$f^{(a)}(\hat{a}^+\hat{a})|m\rangle = |m\rangle \lim_{n\to\infty} \int_{-\infty}^{+\infty} g(\lambda)\, e^{i\lambda} s_{n,m}(i\lambda)\, d\lambda \tag{18}$$

where

$$s_{n,m}(i\lambda) \equiv \sum_{\nu=0}^{n} \binom{m+\nu}{\nu} (1-e^{i\lambda})^\nu = \sum_{\nu=0}^{n} \binom{m+\nu}{m} (1-e^{i\lambda})^\nu$$

$$= \sum_{\nu=0}^{n+m} \binom{\nu}{m} (1-e^{i\lambda})^{\nu-m}. \tag{19}$$

Since for all real λ

$$v \equiv 1 - e^{i\lambda} \neq 1 \tag{20}$$

we obtain

$$s_{n,m}(i\lambda) = \frac{1}{m!}\left(\frac{d^m}{dv^m} \sum_{\nu=0}^{n+m} v^\nu\right)_{v=1-e^{i\lambda}} = \frac{1}{m!}\left(\frac{d^m}{dv^m} \frac{v^{n+m+1}-1}{v-1}\right)_{v=1-e^{i\lambda}} \tag{21}$$

and, therefore,

$$f^{(a)}(\hat{a}^+\hat{a})|m\rangle = \frac{|m\rangle}{m!} \lim_{n\to\infty} \int_{-\infty}^{+\infty} g(\lambda)\, e^{i\lambda}\left(\frac{d^m}{dv^m} \frac{v^{n+m+1}-1}{v-1}\right)_{v=1-e^{i\lambda}} d\lambda. \tag{22}$$

The special case $m = 0$ yields

$$f^{(a)}(\hat{a}^+\hat{a})|0\rangle = |0\rangle \lim_{n\to\infty} \int_{-\infty}^{+\infty} g(\lambda)[1-(1-e^{i\lambda})^{n+1}]\, d\lambda. \tag{22a}$$

Until now we have not been able to decide whether the right-hand side of (22) equals the state $f(m)|m\rangle$ (see (3*b*)) unless we can perform the limit $n \to \infty$. However, the existence and value of the limit depend crucially on the Fourier transform $g(\lambda)$. In § 4 conditions on $g(\lambda)$ are given which ensure the validity of equation (3*b*).

4. Sufficient criteria for antinormal ordering to be consistent

4.1. First criterion

Let us call a normally or antinormally ordered form of $f(\hat{a}^+\hat{a})$ consistent if equation (3*b*) is satisfied. Then, according to § 3, all normally ordered expressions $f^{(n)}(\hat{a}^+\hat{a})$ are consistent.

For the case of antinormal ordering we start from equation (18). Elementary analysis allows us to write

$$f^{(a)}(\hat{a}^+\hat{a})|m\rangle = |m\rangle \int_{-\infty}^{+\infty} g(\lambda)\, e^{i\lambda} \lim_{n\to\infty} s_{n,m}(i\lambda)\, d\lambda \tag{23}$$

if the sequence of partial sums $s_{n,m}(i\lambda)$ converges uniformly, at least for those parts of the λ axis where $g(\lambda)$ does not vanish. Since

$$s_m(i\lambda) \equiv \lim_{n\to\infty} s_{n,m}(i\lambda) = \sum_{\nu=0}^{\infty} \binom{m+\nu}{\nu} v^{\nu} \tag{24}$$

is a power series in v, $s_{n,m}$ is uniformly convergent for all v with $|v| \leq r-\varepsilon$ where r is the radius of convergence and $\varepsilon > 0$ infinitesimal. It has been shown previously (Baltin 1982) that

$$r = 1 \tag{25}$$

and that

$$s_m(i\lambda) = e^{-i\lambda(m+1)} \tag{26}$$

for

$$|1-e^{i\lambda}| < 1. \tag{27}$$

Thus $s_{n,m}(i\lambda)$ is uniformly convergent for $|1-e^{i\lambda}| \leq 1-\varepsilon$ or

$$\cos\lambda \geq \tfrac{1}{2}+\varepsilon \tag{27a}$$

which corresponds to

$$\lambda \in \mathbb{J} \equiv \bigcup_{k=-\infty}^{+\infty} \mathbb{J}_k \tag{28}$$

where $\mathbb{J}_k$ are closed intervals of the real λ axis defined by

$$\mathbb{J}_k \equiv [2\pi k - \tfrac{1}{3}\pi + \varepsilon,\, 2\pi k + \tfrac{1}{3}\pi - \varepsilon], \qquad k = 0, \pm1, \pm2, \ldots. \tag{29}$$

If we suppose $g(\lambda) = 0$ for all λ outside $\mathbb{J}$ the only contributions to the integral (23) come from those parts of the domain of integration where $s_{n,m}$ is uniformly convergent. Consequently, we obtain

$$f^{(a)}(\hat{a}^+\hat{a})|m\rangle = |m\rangle \int_{-\infty}^{+\infty} g(\lambda)\, e^{i\lambda}\, e^{-i\lambda(m+1)}\, d\lambda = f(m)|m\rangle$$

due to equation (10a).

Thus we are able to state the following.

First criterion. If the Fourier transform $g(\lambda)$ of the c-number function $f(x)$ vanishes for all $\lambda \notin \mathbb{J}$ then $f^{(a)}(\hat{a}^+\hat{a})|m\rangle$ is defined for all occupation number states $|m\rangle$ and equals $f(m)|m\rangle$.

4.2. Second criterion

When $g(\lambda) \neq 0$ for some interval with $|1-e^{i\lambda}| \geq 1$ series (24) is divergent so that expression (23) becomes meaningless. In this case interchange of the limiting process with the λ integration is clearly forbidden. However, this does not imply that the right-hand side of equation (18) does not exist. One can realise that parts of the integrand blowing up as $n \to \infty$ will cancel when the integration is performed first so that the limit $n \to \infty$ can be taken subsequently. We are now going to derive a criterion which states sufficient conditions under which this cancellation happens. The mathematical tool of the derivation is complex function theory.

Let the complex function $h(w)$ be analytic on and inside the unit circle, $|w| \leq 1$, of the complex w-plane. Suppose, furthermore,

(1) $g(\lambda)$ coincides with $h(w)$ on the unit circle

$$g(\lambda) = h(e^{i\lambda}) \tag{30}$$

for all real λ with $|\lambda| \leq c$, where c is some positive number;

$$(2) \qquad g(\lambda) = 0 \text{ for } |\lambda| > c. \tag{31}$$

In deriving the criterion we shall proceed in three steps:

(i) transformation of $\int_{-\infty}^{+\infty} g(\lambda)\, e^{i\lambda} s_{n,m}(i\lambda)\, d\lambda$ (see equation (18));

(ii) performing the limit $n \to \infty$;

(iii) transformation of $f(\hat{a}^+\hat{a})|m\rangle = f(m)|m\rangle$ (equation (5)).

After these steps have been done, the relation between $f^{(a)}(\hat{a}^+\hat{a})|m\rangle$ and $f(\hat{a}^+\hat{a})|m\rangle$ is established by comparison and formulated in the criterion at the end of this subsection.

(i) On account of condition (31) we write

$$f^{(a)}(\hat{a}^+\hat{a})|m\rangle = |m\rangle \lim_{n\to\infty} R_{n,m}(c) \tag{18a}$$

where

$$R_{n,m}(c) \equiv \int_{-c}^{+c} g(\lambda)\, e^{i\lambda} s_{n,m}(i\lambda)\, d\lambda \tag{32}$$

can be written as a complex contour integral

$$R_{n,m}(c) = -i \int_L h(w) \sum_{\nu=0}^{n} \binom{m+\nu}{\nu} (1-w)^{\nu}\, dw \tag{33}$$

by substituting $w = e^{i\lambda}$ and the definition (19) of $s_{n,m}$. Let us split up

$$c = c_0 + k\pi \qquad (k \text{ integer} \geq 0) \qquad 0 \leq c_0 < \pi. \tag{34a, b}$$

As λ runs from $-c$ to $+c$, equation (32), w moves along L, namely counter-clockwise on the unit circle from $\exp[i(-c_0 - k\pi)] = (-1)^k \exp(-ic_0)$ to $\exp[i(c_0 + k\pi)] = (-1)^k \exp(ic_0)$ thereby describing a full circle k times. When the paths in the w plane corresponding to $-c \leq \lambda \leq -k\pi$ and to $+k\pi \leq \lambda \leq c$ are denoted by L^- and L^+, respectively, we obtain

$$R_{n,m}(c) = -i\left(\int_{L^-} + k \oint + \int_{L^+}\right). \tag{35}$$

Since the integrand of (33) is analytic for $|w| \leq 1$ and for finite n we obtain

$$\oint h(w) \sum_{\nu=0}^{n} \binom{m+\nu}{\nu} (1-w)^{\nu}\, dw = 0. \tag{36}$$

If $c_0 = 0$, i.e. $c = k\pi$, it follows from (35), (36) immediately that

$$R_{n,m}(k\pi) = 0 \tag{35a}$$

for all n and m and, therefore, in the limit $n \to \infty$

$$f^{(a)}(\hat{a}^+\hat{a})|m\rangle = 0. \tag{37}$$

The case $c_0 > 0$ which is treated in appendix 1 yields for $m > 0$

$$R_{n,m}(c) = \frac{\mathrm{i}}{m!}\left[\left(\sum_{\mu=0}^{m-1}(-1)^{\mu}\frac{\mathrm{d}^{\mu}t(v)}{\mathrm{d}v^{\mu}}\frac{\mathrm{d}^{m-\mu-1}y_{n+m}(v)}{\mathrm{d}v^{m-\mu-1}}\right)_{v=z^*}^{z} + (-1)^m\int_V y_{n+m}(v)\frac{\mathrm{d}^m t(v)}{\mathrm{d}v^m}\,\mathrm{d}v\right] \tag{38}$$

where $z = z_+$ or z_-, equations (A1.7*a*), for k even or odd, respectively. $y_l(v)$ and $t(v)$ are defined by (A1.5) and (A1.6) and the integration is to be performed along the contours $V^{\pm}$ (figures 1(*a*)–1(*b*)) for k even or odd, respectively.

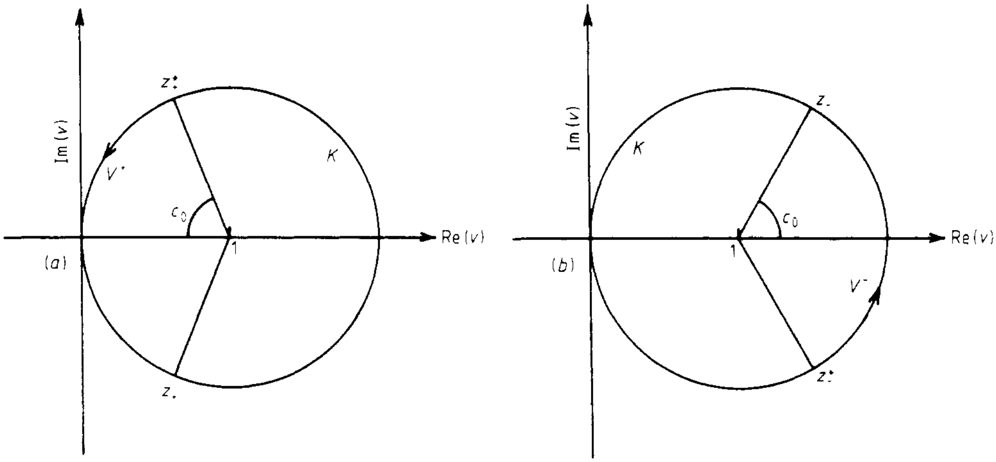

Figure 1. (*a*) Contour V^+ of integration in expressions (38), (38*a*) if $c = k\pi + c_0$ with k even. V^+ extends from z_+^* to z_+ on the unit circle *K*. (*b*) Contour V^- of integration in expressions (38), (38*a*) if $c = k\pi + c_0$ with k odd. V^- extends from z_-^* to z_- on the unit circle *K*.

If $m = 0$, we simply obtain

$$R_{n,0}(c) = \mathrm{i}\int_V t(v)y_n(v)\,\mathrm{d}v. \tag{38a}$$

(ii) The existence of $\lim_{n\to\infty} R_{n,m}(c)$ is ensured if the limit of the sum in equation (38) exists and if the contours $V^{\pm}$ can be deformed such that $|v| \leq 1-\varepsilon$ along the entire path of integration. As is derived in appendix 2, this is satisfied:

(a) for *k even*, if $|z_+| < 1$ or, equivalently,

$$0 \leq c_0 < \tfrac{1}{3}\pi \tag{A2.8a}$$

and the contour V^+ ($\equiv W^+$) remains unchanged.

(b) for k *odd*, if $|z_-|<1$ or

$$\tfrac{2}{3}\pi < c_0 < \pi \tag{A2.8b}$$

and V^- is replaced by W^- (see figure 3).

The limit is given by (A2.5). From (A2.4) and (A2.7) we then obtain

$$f^{(a)}(\hat{a}^+\hat{a})|m\rangle = \frac{\mathrm{i}|m\rangle}{m!}\left[\left(\sum_{\mu=0}^{m-1}(-1)^\mu t^{(\mu)}(v)\frac{(m-\mu-1)!}{(1-v)^{m-\mu}}\right)\right]_{v=z_\pm^*}^{z_\pm} + (-1)^m\int_{W^\pm} t^{(m)}(v)\frac{\mathrm{d}v}{1-v}\Bigg]$$

$$= |m\rangle R_m^\pm. \tag{39}$$

(iii) In order to compare $f^{(a)}(\hat{a}^+\hat{a})|m\rangle$ with $f(m)|m\rangle$ let us set $x=m$ in equation (10a) and substitute $w=\mathrm{e}^{\mathrm{i}\lambda}$.

$$f(m)=\int_{-c}^{+c} g(\lambda)\,\mathrm{e}^{-\mathrm{i}\lambda m}\,\mathrm{d}\lambda = -\mathrm{i}\int_L \frac{h(w)}{w^{m+1}}\,\mathrm{d}w. \tag{40}$$

The integrand has a pole at $w=0$. If we split up the contour integral in exactly the same manner as we did in the case of $f^{(a)}(\hat{a}^+\hat{a})|m\rangle$, we obtain

$$f(m) = -\mathrm{i}\left(\int_{L^-} + k\oint + \int_{L^+}\right) \tag{41}$$

where

$$-\mathrm{i}k\int\frac{h(w)\,\mathrm{d}w}{w^{m+1}} = \frac{2\pi k}{m!}\frac{\mathrm{d}^m h(w)}{\mathrm{d}w^m}\bigg|_{w=0} = (-1)^m\frac{2\pi k}{m!}t^{(m)}(1) \tag{42}$$

due to the Cauchy formula. Furthermore, we obtain

$$-\mathrm{i}\left(\int_{L^-}+\int_{L^+}\right) = (-1)^{km}\int_{-c_0}^{+c_0} h((-1)^k\,\mathrm{e}^{\mathrm{i}\lambda})\,\mathrm{e}^{-\mathrm{i}\lambda m}\,\mathrm{d}\lambda$$

$$= \frac{\mathrm{i}}{m!}\int_V t(v)\frac{\mathrm{d}^m}{\mathrm{d}v^m}\left(\frac{1}{1-v}\right)\mathrm{d}v$$

$$= \frac{\mathrm{i}}{m!}\left[\left(\sum_{\mu=0}^{m-1}(-1)^\mu t^{(\mu)}(v)\left(\frac{1}{1-v}\right)^{(m-\mu-1)}\right)_{v=z^*}^{z} + (-1)^m\int_V t^{(m)}(v)\frac{\mathrm{d}v}{1-v}\right] \tag{43}$$

analogously to equation (39). Again $V=V^+(V^-)$ and $z=z_+(z_-)$ for k even (odd). Since, however, no geometric series occurs in these expressions, there is no problem of convergence that would force us to confine z to satisfy $|z|<1$.

We obtain from equations (41)–(43) and from (A2.5) and (A2.9)

$$f(m) = (-1)^m(2\pi k/m!)t^{(m)}(1) + R_m^+ \qquad k \text{ even} \tag{44a}$$

and

$$f(m) = (-1)^m[2\pi(k+1)/m!]t^{(m)}(1) + R_m^- \qquad k \text{ odd}. \tag{44b}$$

By comparison with equation (39) we obtain for k even

$$f^{(a)}(\hat{a}^+\hat{a})|m\rangle = f(m)|m\rangle - (2\pi k/m!)h^{(m)}(0)|m\rangle \tag{45a}$$

if condition (A2.8*a*) is satisfied. Similarly, for k odd

$$f^{(a)}(\hat{a}^+\hat{a})|m\rangle = f(m)|m\rangle - [2\pi(k+1)/m!]h^{(m)}(0)|m\rangle \tag{45b}$$

if condition (A2.8*b*) or

$$c_0 = 0 \tag{46}$$

is satisfied.

Relations (45*a*, *b*) can be given a more compact form. We find that

$$f^{(a)}(\hat{a}^+\hat{a})|m\rangle = f(m)|m\rangle - (4\pi j/m!)h^{(m)}(0)|m\rangle \tag{45}$$

if

$$2\pi j - \tfrac{1}{3}\pi < c < 2\pi j + \tfrac{1}{3}\pi \tag{47}$$

or if

$$c = (2j-1)\pi \tag{48}$$

for $j = 1, 2, 3, \ldots$. The case $j = 0$ yields

$$f^{(a)}(\hat{a}^+\hat{a})|m\rangle = f(m)|m\rangle \tag{49}$$

if

$$0 \leqslant c < \tfrac{1}{3}\pi. \tag{50}$$

For $j > 0$, relation (49) holds for all m only if

$$h^{(m)}(0) = 0 \tag{51}$$

($m = 0, 1, 2, 3, \ldots$). Since $h(w)$ was assumed to be analytic for $|w| \leqslant 1$, it can be represented by its power series expansion

$$h(w) = \sum_{\nu=0}^{\infty} \frac{w^\nu}{\nu!} h^{(\nu)}(0) \tag{52}$$

in this domain. Therefore, from (51) we obtain $h(w) \equiv 0$ ($|w| \leqslant 1$) which means that $g(\lambda) \equiv 0$ for all λ and thus $f(x) \equiv 0$. It follows that if $g(\lambda)$ satisfies assumptions (30) and (31) with $c \notin [0, \frac{1}{3}\pi)$, equation (49) cannot be valid for all states $|m\rangle$ unless $f(x) \equiv 0$.

The results of this subsection can be stated in the following.

Second criterion. If the Fourier transform $g(\lambda)$ of the c-number function $f(x)$ satisfies

$$g(\lambda) = \begin{cases} h(\mathrm{e}^{\mathrm{i}\lambda}) & \text{for } |\lambda| \leqslant c \text{ where } h(w) \text{ is a function of the complex variable } w \\ & \text{and analytic for } |w| \leqslant 1 \\ 0 & \text{for } |\lambda| > c \end{cases}$$

then

(a) $f^{(a)}(\hat{a}^+\hat{a})|m\rangle$ does not exist for any state $|m\rangle$ if none of the conditions (47), (48), (50) or $c = 2\pi j \pm \frac{1}{3}\pi$ is satisfied. Hence, for this case $\mathbb{D}^{(a)} = \varnothing$. (If $c = 2\pi j \pm \frac{1}{3}\pi$ no general statement can be made. Rather each specific case has to be investigated separately.)

(b) $f^{(a)}(\hat{a}^+\hat{a})|m\rangle$ exists and is given by (45) if (47) or (48) is satisfied. However, (49) cannot be true for all m unless $f(x) \equiv 0$.

(c) $f^{(a)}(\hat{a}^+\hat{a})|m\rangle = 0$ for all m if equation (48) holds.

(d) If c satisfies (50), $f^{(a)}(\hat{a}^+\hat{a})|m\rangle$ exists and equals $f(m)|m\rangle$ for all m, i.e. only in this case is antinormal ordering consistent.

5. Examples

In this section two examples of operator functions $f(\hat{a}^+\hat{a})$ are considered. They are both chosen such that all calculations remain tractable from the mathematical point of view. The criteria derived above are partially applicable to the first example, but not to the second. In the latter case, however, we shall be able to prove by direct methods of evaluation that $f^{(a)}(\hat{a}^+\hat{a})|m\rangle$, at least for $m=0$, does not exist.

5.1. Antinormal ordering of $f(\hat{a}^+\hat{a}) = \sin(c\hat{a}^+\hat{a})/c\hat{a}^+\hat{a}$

The function

$$f(x) \equiv \sin(cx)/cx \qquad c>0 \tag{53}$$

has a very simple Fourier transform (see e.g. Erdélyi *et al* 1954)

$$g(\lambda) = \frac{1}{\pi}\int_0^\infty \cos(\lambda x)\,\frac{\sin(cx)}{cx}\,\mathrm{d}x = \begin{cases} \frac{1}{2}c^{-1} & \text{if } |\lambda|<c \\ \frac{1}{4}c^{-1} & \text{if } |\lambda|=c \\ 0 & \text{if } |\lambda|>c. \end{cases} \tag{54}$$

Let us first calculate $f^{(a)}(\hat{a}^+\hat{a})$ from equation (15*b*):

$$d_\nu = \frac{1}{2c\nu!}\int_{-c}^{+c} \mathrm{e}^{\mathrm{i}\lambda}(1-\mathrm{e}^{\mathrm{i}\lambda})^\nu\,\mathrm{d}\lambda = \frac{\mathrm{i}}{2c(\nu+1)!}[(1-\mathrm{e}^{\mathrm{i}c})^{\nu+1} - (1-\mathrm{e}^{-\mathrm{i}c})^{\nu+1}]$$

or

$$d_\nu = \begin{cases} \dfrac{(-1)^k}{c(2k+1)!}[2\sin(\frac{1}{2}c)]^{2k+1}\cos[\frac{1}{2}c(2k+1)] & \text{for } \nu=2k \quad (k\geqslant 0) \\ \dfrac{(-1)^k}{c(2k)!}[2\sin(\frac{1}{2}c)]^{2k}\sin(ck) & \text{for } \nu=2k-1 \quad (k>0). \end{cases} \tag{55}$$

Evidently, the coefficients of the formal expansion of $f^{(a)}(a^+a)$, equation (14*b*), have no singularities.

When the states $f^{(a)}(\hat{a}^+\hat{a})|m\rangle$ are considered, one has to distinguish four cases:

(1) $0\leqslant c<\frac{1}{3}\pi$ (50)

(2) $|c-2j\pi|<\frac{1}{3}\pi$ (47)

(3) $c=(2j-1)\pi$ (48)

(4) $0<|c-(2j-1)\pi|\leqslant\frac{2}{3}\pi$. (56)

Here $j>0$ is an arbitrary integer. Each non-negative value of c falls into one of these four classes.

Case (1). Both the criteria of § 4 are satisfied. From $c\in\mathbb{J}_0$, equation (29), it follows that $g(\lambda)=0$ for $\lambda\notin\mathbb{J}_0\subset\mathbb{J}$. Thus the first criterion states that

$$\left[\frac{\sin(c\hat{a}^+\hat{a})}{c\hat{a}^+\hat{a}}\right]^{(a)}|m\rangle = \frac{\sin(cm)}{cm}|m\rangle \tag{57}$$

for all $m=0,1,2,\ldots$.

The second criterion can be applied by putting

$$h(w) \equiv \tfrac{1}{2}c^{-1} = \text{constant}. \tag{58}$$

Then (57) follows from part (d) of the criterion.

Case (2). The first criterion cannot be applied since $g(\lambda) = \frac{1}{2}c^{-1} \neq 0$ for all λ with $-c < \lambda < +c$ thereby including intervals $[(2l+1)\pi - \frac{2}{3}\pi,\ (2l+1)\pi + \frac{2}{3}\pi]$ ($l = 0, 1, 2, \ldots, j-1$) sharing no common points with $\mathbb{J}$. According to the second criterion, part (b), and to equation (45), we obtain

$$\left[\frac{\sin(c\hat{a}^+\hat{a})}{c\hat{a}^+\hat{a}}\right]^{(a)}|m\rangle = \left[\frac{\sin(cm)}{cm} - \frac{4\pi j\delta_{m,0}}{m!\,2c}\right]|m\rangle \tag{59}$$

since clearly all derivatives of the constant (58) vanish (except for $h^{(0)} = h$). Thus, for $m > 0$, equation (57) is valid, but for $m = 0$

$$\left[\frac{\sin(c\hat{a}^+\hat{a})}{c\hat{a}^+\hat{a}}\right]^{(a)}|0\rangle = \left(1 - \frac{2\pi j}{c}\right)|0\rangle \neq |0\rangle. \tag{59a}$$

Case (3). Again, only criterion (2) is applicable. According to part (c) we obtain for all $m \geqslant 0$

$$\left\{\frac{\sin[(2j-1)\pi\hat{a}^+\hat{a}]}{(2j-1)\pi\hat{a}^+\hat{a}}\right\}^{(a)}|m\rangle = 0. \tag{60}$$

On the other hand,

$$f(m)|m\rangle = |m\rangle \sin[(2j-1)\pi m]/(2j-1)\pi m = 0 \tag{61}$$

only if $m > 0$; however,

$$f(0)|0\rangle = |0\rangle \neq 0. \tag{61a}$$

Therefore, as for case (2), antinormal ordering is consistent except for $m = 0$.

Case (4). The second criterion, part (a), states that $f^{(a)}(\hat{a}^+\hat{a})|m\rangle$ does not exist if

$$0 < |c - (2j-1)\pi| < \tfrac{2}{3}\pi.$$

We still have to treat the cases

$$c = (2j-1)\pi \pm \tfrac{2}{3}\pi$$

which correspond to

$$c = 2j\pi - \tfrac{1}{3}\pi \tag{62a}$$

and

$$c = 2(j-1)\pi + \tfrac{1}{3}\pi \tag{62b}$$

respectively. For the sake of simplicity, let us confine discussion to states $|0\rangle$ and $|1\rangle$ only.

5.1.1. $m=0$. When c satisfies ($62a$) this corresponds to $c_0=\frac{2}{3}\pi$ and $k=2j-1$ (compare equations ($34a, b$)) and from ($38a$), (58), (A1.5), (A1.6) and (A1.$7b$), we obtain

$$R_{n,0}=\frac{\mathrm{i}}{2c}\int_{V^-}\sum_{\nu=0}^{n}v^\nu\,\mathrm{d}v=\frac{\mathrm{i}}{2c}\sum_{\nu=0}^{n}\frac{z_-^{\nu+1}-z_+^{*\nu+1}}{\nu+1}.$$

Noting

$$z_-=1+\exp(\tfrac{2}{3}\mathrm{i}\pi)=\exp(\tfrac{1}{3}\mathrm{i}\pi) \tag{63}$$

we find

$$R_{n,0}=-\frac{1}{c}\sum_{\nu=0}^{n}\frac{\sin[\frac{1}{3}\pi(\nu+1)]}{\nu+1}. \tag{64}$$

The limit $n\to\infty$ exists and can be expressed by the Bernoulli polynomial $B_1(x)=x-\frac{1}{2}$ (Abramowitz and Stegun 1972)

$$\sum_{\nu=0}^{\infty}\frac{\sin[\frac{1}{3}\pi(\nu+1)]}{\nu+1}=-\pi B_1(\tfrac{1}{6})=\tfrac{1}{3}\pi, \tag{65}$$

hence, according to ($18a$)

$$f^{(\mathrm{a})}(\hat{a}^+\hat{a})|0\rangle=|0\rangle\lim_{n\to\infty}R_{n,0}=-\frac{\pi}{3c}|0\rangle=\frac{1}{1-6j}|0\rangle \tag{66}$$

which equals expression ($59a$) for $c\to[2j\pi-\frac{1}{3}\pi]_+$. When c is given by ($62b$) we get $c_0=\frac{1}{3}\pi$ and $k=2j-2$ even. Thus, we have to evaluate ($38a$) with $V=V^+$ and

$$z_+=1-\exp(\tfrac{1}{3}\mathrm{i}\pi)=\exp(-\tfrac{1}{3}\mathrm{i}\pi)=z_-^*. \tag{67}$$

The calculation being quite similar to the case just treated leads to

$$f^{(\mathrm{a})}(\hat{a}^+\hat{a})|0\rangle=(6j-5)^{-1}|0\rangle \tag{68}$$

which again is in agreement with ($59a$) for $c\to[2(j-1)\pi+\frac{1}{3}\pi]_-$.

5.1.2. $m=1$. Since all derivatives of $t(v)$ vanish identically, only the term with $\mu=0$ contributes to expression (38), so we are left with

$$R_{n,1}=\frac{\mathrm{i}}{2c}[y_{n+1}(z)-y_{n+1}(z^*)]=\frac{\mathrm{i}}{2c}\left(\frac{z^{n+2}-1}{z-1}-\frac{z^{*n+2}-1}{z^*-1}\right). \tag{69}$$

When c is given by ($62a$), insertion of $z=z_-$ (equation (63)) leads after some algebra to

$$R_{n,1}=-c^{-1}[\sin(\tfrac{1}{3}\pi)+\sin(\tfrac{1}{3}n\pi)]. \tag{70}$$

Evidently, $\lim_{n\to\infty}R_{n,1}$ does not exist. It follows immediately from (67), (69) that the limit of $R_{n,1}$ does not exist either if c satisfies ($62b$).

The results of this subsection are shown in figures 2(a) and (b) where the ratio

$$\beta\equiv\frac{\|f^{(\mathrm{a})}(\hat{a}^+\hat{a})|m\rangle\|}{\|f(\hat{a}^+\hat{a})|m\rangle\|}=\frac{\lim_{n\to\infty}R_{n,m}}{f(m)} \tag{71}$$

is drawn against c. When $\lim_{n\to\infty}R_{n,m}=f(m)=0$ we have set $\beta\equiv1$.

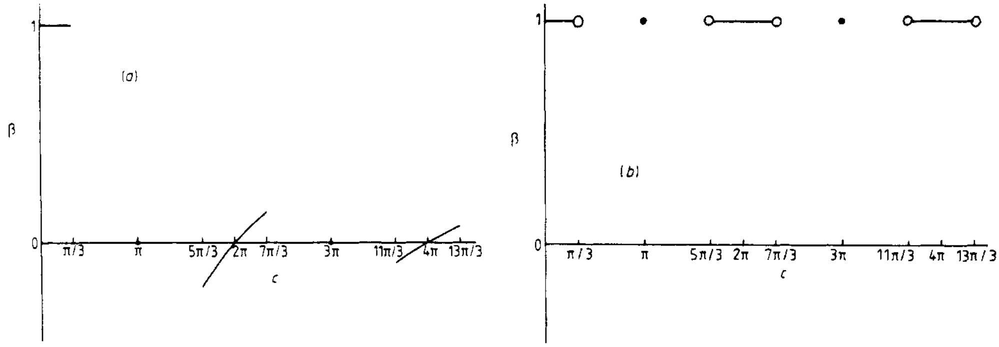

Figure 2. (a) Ratio β, equation (71), against c for $m=0$. At the empty parts of the diagram $\lim_{n\to\infty} R_{n,0}$ and hence β does not exist. Note the isolated points at $c=\pi, 3\pi, \ldots$. (b) As for (a) for $m>0$. Light circles denote points to be excluded from $\beta(c)$. At the points $c=\pi, 3\pi, \ldots$ where both $\lim_{n\to\infty} R_{n,m}$ and $f(m)$ vanish we have set $\beta=1$.

5.2. *Antinormal ordering of* $f(\hat{a}^+\hat{a})=[b^2+(\hat{a}^+\hat{a})^2]^{-1}$ $(b>0)$

Since the power-series expansion of $f(x)=(b^2+x^2)^{-1}$ $(b>0)$ about the origin

$$f(x)=b^{-2}[1-(x/b)^2+(x/b)^4\mp\ldots] \tag{72}$$

converges for $|x|<b$ the corresponding operator expansion with x replaced by $\hat{a}^+\hat{a}$ can be applied at least to state $|0\rangle$ yielding

$$f(\hat{a}^+\hat{a})|0\rangle=b^{-2}|0\rangle. \tag{73}$$

Let us now establish the formal antinormal-ordering expansion (14b), (15b) and then let this expansion act on state $|0\rangle$. The Fourier transform (10a) is given by (Erdélyi *et al* 1954)

$$g(\lambda)=\frac{1}{\pi}\int_0^\infty f(x)\cos(\lambda x)\,\mathrm{d}x=\frac{1}{2b}\mathrm{e}^{-b|\lambda|} \tag{74}$$

and, therefore, the coefficients d_ν

$$d_\nu=\frac{1}{2b\nu!}\int_{-\infty}^{+\infty}\mathrm{e}^{-b|\lambda|}\mathrm{e}^{\mathrm{i}\lambda}(1-\mathrm{e}^{\mathrm{i}\lambda})^\nu\,\mathrm{d}\lambda=\frac{1}{\nu!}(g_\nu-g_{\nu+1}) \tag{75}$$

where

$$g_\nu\equiv\int_{-\infty}^{+\infty}g(\lambda)(1-\mathrm{e}^{\mathrm{i}\lambda})^\nu\,\mathrm{d}\lambda. \tag{76}$$

Since here $g(\lambda)$ is an even function of λ we have

$$g_\nu=\frac{1}{2b}\int_0^\infty\mathrm{e}^{-b\lambda}[(1-\mathrm{e}^{\mathrm{i}\lambda})^\nu+(1-\mathrm{e}^{-\mathrm{i}\lambda})^\nu]\,\mathrm{d}\lambda. \tag{77}$$

g_ν and d_ν have been evaluated in appendix 3. We thus obtain from (14b), (A3.8)

$$\left[\frac{1}{b^2+(\hat{a}^+\hat{a})^2}\right]^{(\mathrm{a})}=-\frac{1}{b}\sum_{\nu=0}^{\infty}\mathrm{Im}\left(\frac{\Gamma(1+\mathrm{i}b)}{\Gamma(\nu+2+\mathrm{i}b)}\right)\hat{a}^\nu\hat{a}^{+\nu}. \tag{78}$$

The action of this operator expansion upon the state $|0\rangle$ is given by the limit (22a).

Since $g(\lambda) \neq 0$ along the entire real λ axis the first criterion (§ 4.1) fails. Furthermore, $g(\lambda)$ cannot be represented as an analytic function of $w = e^{i\lambda}$, not only because of the appearance of the absolute value of λ in the exponent. For example, even a function

$$e^{-b\lambda} = \exp(ib \ln w)$$

is not analytic. Therefore, the second criterion (§ 4.2) cannot be applied either. We are thus forced to evaluate (22a) directly.

The limit in (22a) can be written as

$$\lim_{\nu\to\infty}\left(\int_{-\infty}^{+\infty} g(\lambda)\,d\lambda - g_{\nu+1}\right) = \frac{1}{b^2} - \lim_{\nu\to\infty} g_\nu. \tag{79}$$

The limit of g_ν, however, does not exist as has been shown in appendix 3 (equation (A3.20)). Thus

$$[1/[b^2 + (\hat{a}^+\hat{a})^2]]^{(a)}|0\rangle$$

does not exist either.

6. Conclusion

When the antinormally ordered expansion $f^{(a)}(\hat{a}, \hat{a}^+)$ of an operator function $f(\hat{a}, \hat{a}^+)$ is applied to a state $|\psi\rangle$ belonging to the domain of definition of f the following results are possible:

(i) $f^{(a)}|\psi\rangle$ does not exist;
(ii) $f^{(a)}|\psi\rangle$ exists, but $f^{(a)}|\psi\rangle \neq f|\psi\rangle$;
(iii) $f^{(a)}|\psi\rangle = f|\psi\rangle$.

This is in contrast to the action of the normally ordered expansion $f^{(n)}$ upon $|\psi\rangle$ where $f^{(n)}|\psi\rangle = f|\psi\rangle$ holds always.

Since the peculiar behaviour of $f^{(a)}|\psi\rangle$ can be observed even for fairly simple operators f, we have considered the special class of functions which depend on the number operator $\hat{a}^+\hat{a}$ but not on $\hat{a}$ and $\hat{a}^+$ separately. In that way, all calculations become easier by use of the eigenstate representation $\{|m\rangle\}$ of $\hat{a}^+\hat{a}$.

The normal- and antinormal-ordering expansions of $f(\hat{a}^+\hat{a})$ have been derived in terms of the Fourier transform $g(\lambda)$ of the associated c-number function $f(x)$ (see equations (14a, b), (15a, b)). When the states $|\psi\rangle$ are chosen to be the eigenstates $|m\rangle$ the action of $f^{(n)}$ on $|m\rangle$ is given by $f(m)|m\rangle$. However, $f^{(a)}|m\rangle$ reproduces $|m\rangle$ up to a factor which is the limit of a sequence of integrals involving $g(\lambda)$ (equations (22), (22a)). The main part of this paper deals with the question whether this limit exists and, if so, what value it takes on. If the limit exists for all m and has the value $f(m)$ then we called the antinormal-ordering expansion 'consistent'.

The difficulties we encounter with the convergence of $f^{(a)}|m\rangle$ are not too surprising since they can be traced back to a phenomenon well known for a very long time from the theory of conditionally convergent series. The sum of a series of this kind can change its value or can even be made divergent by rearrangement of infinitely many members. This remains true *a fortiori* when an infinite number of new terms is added and, simultaneously, subtracted, however, in a different order. It is just this kind of procedure involved when a power series expansion of $f(\hat{a}^+\hat{a})$ is transformed into $f^{(a)}(\hat{a}^+\hat{a})$ by repeated use of the commutation relation between $\hat{a}$ and $\hat{a}^+$ and subsequently is applied to $|m\rangle$.

In § 4 conditions on $g(\lambda)$ are given which are sufficient that the limit (22) exists and has the value $f(m)|m\rangle$.

The first criterion (§ 4.1) states that consistent antinormal ordering is possible if $g(\lambda)$ vanishes outside the union $\mathbb{J}$ (equations (28), (29)) of intervals of the λ axis. The derivation is based on uniform convergence of the integrand of (22) or (23) inside $\mathbb{J}$ which allows interchange of the limiting process with integration.

The second criterion can be applied provided $g(\lambda)$ can be written as an analytical function of $\exp(\mathrm{i}\lambda)$ inside a certain interval $[-c, +c]$ and being zero outside. Whether $f^{(\mathrm{a})}|m\rangle$ satisfies cases (i), (ii), or (iii) then depends crucially on the extent of this interval. Only if $0 \leq c < \frac{1}{3}\pi$ is $f^{(\mathrm{a})}|m\rangle$ consistent (case (iii)). For other values of c cases (i) and (ii) hold true (see the precise statements at the end of § 4.2). The second criterion is based on the cancellation of parts of the integrand of (22) which tend to grow up as $n \to \infty$. This is possible due to the special property of analyticity imposed on $g(\lambda)$.

In both these criteria rather restrictive conditions have to be satisfied by $g(\lambda)$ to ensure consistent antinormal ordering: the first criterion ensures consistency if $g(\lambda) = 0$ for all λ with $\cos\lambda < \frac{1}{2}$, the second even only if $g(\lambda) = 0$ for $|\lambda| > \frac{1}{3}\pi$.

For instance, all polynomials of $\hat{a}^+\hat{a}$ can be cast into consistent antinormal order because the Fourier transform of a polynomial is given by a finite linear combination of a δ distribution and its derivatives located at $\lambda = 0$, i.e. $g(\lambda) = 0$ for $\lambda \neq 0$. This result is expected, of course, since antinormal order is obtained from the original operator form by a finite number of commutations and rearrangements.

An application of the criteria which is much less trivial has been given by the first example $f(x) = \sin(cx)/cx$, $c > 0$ (see § 5.1). From figures 2(*a*), (*b*) it is seen that all of the three possible cases (i)–(iii) really occur.

Both the criteria cannot be applied to large classes of functions the Fourier transforms of which do not vanish except for isolated points of the λ axis. Nevertheless, from the derivation of the criteria it seems doubtful whether a consistent antinormal ordering exists for these operator functions at all. An example is given by $f(\hat{a}^+\hat{a}) = [b^2 + (\hat{a}^+\hat{a})^2]^{-1}$ (§ 5.2) where expression (22*a*) can be performed exactly, the result being that $f^{(\mathrm{a})}|0\rangle$ does not exist for any value of b. A numerical study of (22*a*) for the operator $f(\hat{a}^+\hat{a}) = \exp[-b(\hat{a}^+\hat{a})^2]$ indicates that (22*a*) for this case does not exist either.

In summary, we can conclude that the formal procedure of antinormal ordering of an operator function $f(\hat{a}^+\hat{a})$ produces an operator expression $f^{(\mathrm{a})}(\hat{a}^+\hat{a})$ which, for large classes of functions, exhibits properties quite different from those of $f(\hat{a}^+\hat{a})$ although the commutation relation $[\hat{a}, \hat{a}^+] = 1$ is taken into account strictly.

Acknowledgments

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Appendix 1. Transformation of $R_{n,m}(c)$ (see (33), (35)) for $c_0 > 0$

Due to equation (36) we obtain from equation (35)

$$R_{n,m}(c) = -\mathrm{i}\left(\int_{L^-} f_{n,m}(w)\,\mathrm{d}w + \int_{L^-} f_{n,m}(w)\,\mathrm{d}w\right) \tag{A1.1}$$

where

$$f_{n,m}(w) \equiv h(w) \sum_{\nu=0}^{n} \binom{m+\nu}{\nu} (1-w)^{\nu}. \tag{A1.2}$$

When $w = \exp[\mathrm{i}(\sigma \pm k\pi)]$ is substituted in $\int_{L^{\pm}}$, respectively, we find

$$R_{n,m}(c) = R_{n,m}(c_0 + k\pi) = (-1)^k \int_{-c_0}^{+c_0} \mathrm{e}^{\mathrm{i}\sigma} f_{n,m}((-1)^k\, \mathrm{e}^{\mathrm{i}\sigma})\, \mathrm{d}\sigma.$$

Introducing a new complex variable

$$v \equiv 1 + (-1)^{k+1}\, \mathrm{e}^{\mathrm{i}\sigma} \tag{A1.3}$$

the integral becomes

$$R_{n,m}(c) = \mathrm{i} \int_V h(1-v) \sum_{v=0}^{n} \binom{m+\nu}{\nu} v^{\nu}\, \mathrm{d}v = \frac{\mathrm{i}}{m!} \int_V t(v) \frac{\mathrm{d}^m}{\mathrm{d}v^m} [y_{n+m}(v)]\, \mathrm{d}v \tag{A1.4}$$

where

$$y_l(v) \equiv \sum_{\nu=0}^{l} v^{\nu} \tag{A1.5}$$

$$t(v) \equiv h(1-v). \tag{A1.6}$$

$t(v)$ is analytic for $|1-v| \leqslant 1$, i.e. on and inside the unit circle K centred about $(1, 0)$ in the v plane. The contour V depends on whether k is even $(V \equiv V^+)$ or odd $(V \equiv V^-)$. Both cases are shown in figures 1(a)–(b) respectively. $V^{\pm}$ lies on the segment of K between

$$z_{\pm} \equiv 1 \mp \mathrm{e}^{\mathrm{i}c_0} \qquad \text{and} \qquad z_{\pm}^{*} = 1 \mp \mathrm{e}^{-\mathrm{i}c_0}. \tag{A1.7a, b}$$

$V^{\pm}$ begins at $z_{\pm}^{*}$, is directed counter-clockwise and ends at $z_{\pm}$. Let us integrate expression (A1.4) m times by parts. This is allowed because $t(v)$, being analytic for $|1-v| \leqslant 1$, has differential quotients of arbitrary order in this domain. We obtain for $m > 0$

$$R_{n,m}^{\pm}(c) = \frac{\mathrm{i}}{m!} \Bigg[\left(\sum_{\mu=0}^{m-1} (-1)^{\mu} \frac{\mathrm{d}^{\mu} t(v)}{\mathrm{d}v^{\mu}} \frac{\mathrm{d}^{m-\mu-1} y_{n+m}(v)}{\mathrm{d}v^{m-\mu-1}} \right)_{v=z_{\pm}^{*}}^{z_{\pm}} + (-1)^m \int_{V^{\pm}} y_{n+m}(v) \frac{\mathrm{d}^m t(v)}{\mathrm{d}v^m}\, \mathrm{d}v \Bigg] \tag{A1.8}$$

where the upper (lower) sign holds for k even (odd).

The case $m = 0$ simply yields from (A1.4)

$$R_{n,0}^{\pm}(c) = \mathrm{i} \int_{V^{\pm}} t(v) y_n(v)\, \mathrm{d}v. \tag{A1.8a}$$

Appendix 2. $R_{n,m}(c)$ in the limit $n \to \infty$

When n tends to infinity $y_{n+m}(v)$ in expression (A1.8) becomes

$$\lim_{n\to\infty} y_{n+m}(v) = (1-v)^{-1} \tag{A2.1}$$

provided that $|v|<1$. Since, according to a well known theorem of analysis, a convergent power-series can be differentiated termwise, we find

$$\lim_{n\to\infty}\left(\frac{\mathrm{d}^{m-\mu-1}y_{n+m}(v)}{\mathrm{d}v^{m-\mu-1}}\right)=\frac{\mathrm{d}^{m-\mu-1}}{\mathrm{d}v^{m-\mu-1}}\left(\frac{1}{1-v}\right)=\frac{(m-\mu-1)!}{(1-v)^{m-\mu}}. \tag{A2.2}$$

Thus, if $|z_\pm|<1$ the limit $n\to\infty$ of the sum in expression (A1.8) exists.

Concerning the integral in (A1.8), we have to distinguish whether k is even or odd. If k is even (contour V^+, see figure 1(a)) and if

$$|v|\leqslant|z_+|<1 \tag{A2.3a}$$

then $y_{n+m}(v)$ converges uniformly for all $v\in V^+$. Therefore, the limit $n\to\infty$ can be performed under the integral sign and we find from (A1.8), (A2.1), and (A2.2)

$$f^{(a)}(\hat{a}^+\hat{a})|m\rangle=|m\rangle\lim_{n\to\infty}R^+_{n,m}(c)\equiv|m\rangle R^+_m \tag{A2.4a}$$

where

$$R^+_m=\frac{\mathrm{i}}{m!}\left[\left(\sum_{\mu=0}^{m-1}(-1)^\mu t^{(\mu)}(v)\frac{(m-\mu-1)!}{(1-v)^{m-\mu}}\right)_{v=z_+^*}^{z_+}+(-1)^m\int_{V^+}t^{(m)}(v)\frac{\mathrm{d}v}{1-v}\right] \tag{A2.5a}$$

if k is even and $|z_+|<1$.

On the other hand, if k is odd (contour V^-, figure 1(b)), condition $|z_-|<1$ does not imply $|v|<1$; e.g. V^- intersects the real v axis at $v=2$. Since, however, the integrand of expression (A1.8) is analytic for $|v-1|\leqslant 1$, V^- can be replaced by a new contour W^- where v moves clockwise on K from z_-^* to z_- (see figure 3). Now, for all $v\in W^-$,

$$|v|\leqslant|z_-|<1 \tag{A2.3b}$$

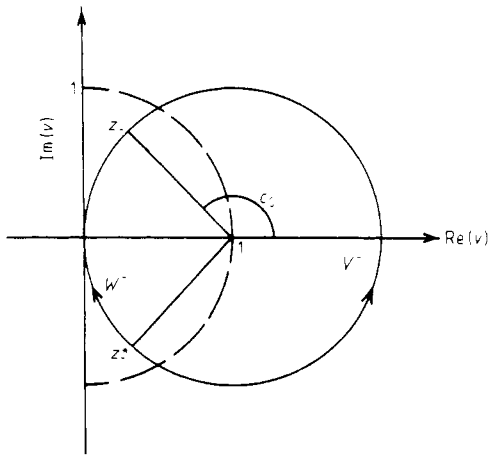

Figure 3. Contours V^- and W^- of integration in expression (A2.5b) if $c=k\pi+c_0$ with k odd. When z_- lies inside the unit circle about the origin (broken line) then $|z_-|<1$, i.e., $c_0>\frac{2}{3}\pi$. This case is shown here.

so that again the limit $n \to \infty$ can be interchanged with the integration along W^-. We obtain

$$f^{(a)}(\hat{a}^+\hat{a})|m\rangle = |m\rangle R_m^- \tag{A2.4b}$$

where now

$$R_m^- = \frac{i}{m!}\left[\left(\sum_{\mu=0}^{m-1}(-1)^\mu t^{(\mu)}(v)\frac{(m-\mu-1)!}{(1-v)^{m-\mu}}\right)_{v=z_-^*}^{z_-} + (-1)^m \int_{W^-} t^{(m)}(v)\frac{dv}{1-v}\right] \tag{A2.5b}$$

if k is odd and $|z_-| < 1$.

When we define paths of integration

$$W = \begin{cases} W^+ \equiv V^+ & k \text{ even} \\ W^- \; (\neq V^-) & k \text{ odd} \end{cases} \tag{A2.6a, b}$$

then equations (A2.4a, b), (A2.5a, b) can be written

$$f^{(a)}(\hat{a}^+\hat{a})|m\rangle = |m\rangle R_m^\pm \tag{A2.4}$$

where

$$R_m^\pm = \frac{i}{m!}\left[\left(\sum_{\mu=0}^{m-1}(-1)^\mu t^{(\mu)}(v)\frac{(m-\mu-1)!}{(1-v)^{m-\mu}}\right)_{v=z_\pm^*}^{z_\pm} + (-1)^m \int_{W^\pm} t^{(m)}(v)\frac{dv}{1-v}\right] \tag{A2.5}$$

if

$$|z_\pm| < 1 \tag{A2.7}$$

for k even or odd, respectively.

$|z_+| < 1$ is satisfied for

$$0 \leq c_0 < \tfrac{1}{3}\pi \tag{A2.8a}$$

and $|z_-| < 1$ implies

$$\tfrac{2}{3}\pi < c_0 < \pi. \tag{A2.8b}$$

Note that after the limit $n \to \infty$ has been taken, the Cauchy formula evidently yields

$$\int_{V^-} t^{(m)}(v)\frac{dv}{1-v} - \int_{W^-} t^{(m)}(v)\frac{dv}{1-v} = \oint t^{(m)}(v)\frac{dv}{1-v} = -2\pi i t^{(m)}(1). \tag{A2.9}$$

If $|z_\pm| > 1$, $\lim_{n\to\infty} R_{n,m}^\pm$ and, therefore, $f^{(a)}(\hat{a}^+\hat{a})|m\rangle$ does not exist, except for $z_- = 2$ which corresponds to $c = k\pi$ for k odd. In this latter case $R_{n,m} = 0$ (see equation (35a)) and hence $R_m^- = 0$. If $|z_\pm| = 1$ no general statement on convergence of $\lim_{n\to\infty} R_{n,m}$ is possible.

Appendix 3. The integral $g_\nu = (2b)^{-1}\int_{-\infty}^{+\infty} e^{-b|\lambda|}(1-e^{i\lambda})^\nu \, d\lambda$ (equations (76), (77))

Using the identity

$$1 - e^{i\lambda} = 2\sin(\tfrac{1}{2}\lambda)\exp[\tfrac{1}{2}i(\lambda - \pi)] \tag{A3.1}$$

we can write the first term of integral (77)

$$\frac{1}{2b}\int_0^\infty e^{-b\lambda}(1-e^{i\lambda})^\nu\, d\lambda$$

$$=\frac{1}{2b}\sum_{l=0}^{\infty}\int_{2\pi l}^{2\pi(l+1)} e^{-b\lambda}(1-e^{i\lambda})^\nu\, d\lambda$$

$$=\frac{2^{\nu-1}}{b}\left(\sum_{l=0}^{\infty} e^{-2\pi b l}\right)\int_0^{2\pi} e^{-b\lambda}\sin^\nu(\tfrac{1}{2}\lambda)\exp[\tfrac{1}{2}i\nu(\lambda-\pi)]\, d\lambda. \qquad \text{(A3.2)}$$

Substitution $\phi=\frac{1}{2}(\lambda-\pi)$ yields

$$\frac{1}{2b}\int_0^\infty e^{-b\lambda}(1-e^{i\lambda})^\nu\, d\lambda=\frac{2^\nu e^{-b\pi}}{b(1-e^{-2b\pi})}\int_{-\pi/2}^{+\pi/2}\exp[(i\nu-2b)\phi]\cos^\nu\phi\, d\phi$$

$$=\frac{2^\nu}{b\sinh(b\pi)}\int_0^{\pi/2}\cos[(\nu+2bi)\phi]\cos^\nu\phi\, d\phi. \qquad \text{(A3.3)}$$

The integral (A3.3) can be solved in terms of the beta function $B(p,q)$ (Gradshteyn and Ryzhik 1965). We obtain

$$\frac{1}{2b}\int_0^\infty e^{-b\lambda}(1-e^{i\lambda})^\nu\, d\lambda=\frac{\pi}{2b\sinh(b\pi)B(\nu+1+ib,\,1-ib)}. \qquad \text{(A3.4)}$$

When B is expressed by the Γ function and the relation

$$\Gamma(1+ib)\Gamma(1-ib)=\pi b/\sinh(\pi b) \qquad \text{(A3.5)}$$

is used we get

$$\frac{1}{2b}\int_0^\infty e^{-b\lambda}(1-e^{i\lambda})^\nu\, d\lambda=\frac{\nu!}{2b^2(\nu+ib)(\nu-1+ib)\ldots(1+ib)}. \qquad \text{(A3.6)}$$

Since the second term of the integral (77) is the complex conjugate to the first we obtain its value immediately by taking the conjugate of the right-hand side of equation (A3.6). So we find

$$g_\nu=\frac{\nu!}{2b^2}\left(\frac{\Gamma(1+ib)}{\Gamma(\nu+1+ib)}+\frac{\Gamma(1-ib)}{\Gamma(\nu+1-ib)}\right) \qquad \text{(A3.7)}$$

and from (75) after some algebra

$$d_\nu=-b^{-1}\,\mathrm{Im}[\Gamma(1+ib)/\Gamma(\nu+2+ib)]. \qquad \text{(A3.8)}$$

To perform the limit (79) we need g_ν for $\nu\gg1$. From the asymptotic representation

$$\Gamma(z)\sim\exp[(z-\tfrac{1}{2})\ln z-z+\tfrac{1}{2}\ln(2\pi)] \qquad \text{(A3.9)}$$

(Abramowitz and Stegun 1972) where $z\to\infty$ is a complex number with $|\arg z|<\pi$, we find for

$$z=\nu+1+ib \qquad \text{(A3.10)}$$

that

$$\ln z=\ln|z|+i\arg(z)=\ln[(\nu+1)^2+b^2]^{1/2}+i\tan^{-1}[b/(\nu+1)] \qquad \text{(A3.11)}$$

and

$$\Gamma(\nu+1+\mathrm{i}b)\sim\sqrt{2\pi}\exp[u_\nu(b)+\mathrm{i}v_\nu(b)] \tag{A3.12}$$

where

$$u_\nu(b)\equiv(\nu+\tfrac{1}{2})\ln[(\nu+1)^2+b^2]^{1/2}-b\tan^{-1}[b/(\nu+1)]-\nu-1 \tag{A3.13}$$

and

$$v_\nu(b)\equiv b\ln[(\nu+1)^2+b^2]^{1/2}+(\nu+\tfrac{1}{2})\tan^{-1}[b/(\nu+1)]-b. \tag{A3.14}$$

Thus we get

$$\Gamma(\nu+1)/\Gamma(\nu+1+\mathrm{i}b)\sim\exp[u_\nu(0)-u_\nu(b)-\mathrm{i}v_\nu(b)] \tag{A3.15}$$

noting that $v_\nu(0)=0$. Since $\nu\gg b$

$$\begin{aligned}u_\nu(0)-u_\nu(b)&\\&=-\tfrac{1}{2}(\nu+\tfrac{1}{2})\ln\left[1+\left(\frac{b}{\nu+1}\right)^2\right]-b\tan^{-1}\left(\frac{b}{\nu+1}\right)\\&\approx-\frac{(\nu+\frac{1}{2})b^2}{2(\nu+1)^2}-\frac{b^2}{\nu+1}=\mathrm{O}(\nu^{-1})\end{aligned}$$

and, therefore,

$$\lim_{\nu\to\infty}\exp[u_\nu(0)-u_\nu(b)]=1. \tag{A3.16}$$

However, for $\nu\to\infty$

$$v_\nu(b)\sim b\ln(\nu+1) \tag{A3.17}$$

and hence

$$\Gamma(\nu+1)/\Gamma(\nu+1+\mathrm{i}b)\sim\exp(-\mathrm{i}b\ln(\nu+1)). \tag{A3.18}$$

Using

$$\Gamma(1\pm\mathrm{i}b)=(\pi b/\sinh(\pi b))^{1/2}\exp[\pm\mathrm{i}\arg\Gamma(1+\mathrm{i}b)] \tag{A3.19}$$

we finally obtain from (A3.7), (A3.18), and (A3.19)

$$g_\nu\sim\frac{1}{b^2}(\pi b/\sinh(\pi b))^{1/2}\cos[\arg\Gamma(1+\mathrm{i}b)-b\ln(\nu+1)] \tag{A3.20}$$

which shows that $\lim_{\nu\to\infty}g_\nu$ does not exist.

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